Subsets of Unit Interval Parking Functions
Enumerated by Fubini Numbers

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Parking Functions

Parking functions were first introduced by Konheim and Weiss in 1966 in the context of hashing problems. They can be defined as n-tuples \( a = (a_1, a_2, \ldots, a_n) \in \mathbb{N}^n \) such that at least i entries of a are at most i for all \( i \in [n] \). As the name suggests, parking functions can also be defined in terms of cars parking in the following arrangement:

Suppose \( n \) cars labeled \( c_1, \ldots, c_n \), enter an empty linear parking lot with \( n \) spaces. Each car \( c_i \) has a preferred space \( a_i \in [n] \). One by one and in order, each car first navigates to its preferred space and parks there if it is empty. If that space is already occupied, it parks in the next available spot, if one exists. If all \( n \) cars can park by this method, the \( n \)-tuple of the cars’ preferences, \( (a_1, \ldots, a_n) \), is called a parking function of length \( n \).

Denote by \( \text{PF}_n \) the set of all parking functions of length \( n \).

Additional Definitions

- Let \( \alpha = (a_1, \ldots, a_n) \) be a parking function. If car \( i \) parks in spot \( a_i \), we call \( a_i - a_{i-1} \) the displacement of car \( i \).
- A unit interval parking function is a parking function in which the individual displacement of each car is at most 1. We refer to the set of unit interval parking functions of length \( n \) as \( \text{UPF}_n \).

\[
\begin{align*}
(2, 1, 2, 3) & \quad \text{is a parking function: } \\
\{1 \leq a_i \leq 3 \} \quad \text{and not a parking function: }
\end{align*}
\]

\[
\begin{align*}
(2, 2, 1) & \quad \text{is a unit interval parking function: } \\
\{1 \leq a_i \leq 3 \} \\
(1, 1, 1) & \quad \text{is not a unit interval parking function: } \\
\{1 \leq a_i \leq 3 \}
\end{align*}
\]

- Let \( (a_1, \ldots, a_n) \) be a parking function. If \( a_i = a_{i+1} \), we call index \( i \) a tie. If \( a_i < a_{i+1} \), index \( i \) is an ascent. If \( a_i > a_{i+1} \), index \( i \) is a descent.

\[
\begin{align*}
\text{Index } 1 \text{ is an ascent in the parking function } (1, 3, 2, 2) \\
\text{Index } 2 \text{ is a descent in the parking function } (1, 3, 2, 2) \\
\text{Index } 3 \text{ is a tie in the parking function } (1, 3, 2, 2)
\end{align*}
\]

Fubini Rankings and Fubini Numbers

A Fubini ranking is a sequence \( \beta = (b_1, \ldots, b_n) \) that represents one way \( n \) players can rank in a competition, allowing ties. [1]. Formally, an \( n \)-tuple \( \beta = (b_1, \ldots, b_n) \in [n]^n \) is a Fubini ranking of length \( n \) if \( b_i \in \mathbb{Z} \) for all \( i \in [n] \) and for all \( i \in [n] \), if \( b_i \in \mathbb{Z} \) and entries of \( \beta \) are equal to \( i \), then the next largest index is \( i + k \). We denote \( \text{FR}_n \) as the set of all Fubini rankings of length \( n \).

The Fubini numbers, denoted \( F_n \), enumerate Fubini rankings: \( F_n = |\text{FR}_n| \).

- \((1, 1, 1)\) is a Fubini ranking: all three players tied at rank 1.
- \((1, 1, 2)\) is not a Fubini ranking: two players tied at rank 1, so no one should place 2.

Enumerations of Parking Functions and Unit Interval Parking Functions

Prior to this project, it was known that:
- The number of parking functions of length \( n \) is \( (n+1)^{n-1} \) [2].
- The weakly increasing (and weakly decreasing) parking functions are enumerated by the Catalan numbers [3].

We aimed to find similar enumerations for unit interval parking functions. In an unpublished paper, Hadaway (2022) found that unit interval parking functions are enumerated by the Fubini Numbers, to which we came up with an independent proof.

\[
\beta = (4, 3, 1, 2, 4, 8, 4, 1) \quad \rightarrow \quad \varphi(\beta) = (4, 3, 1, 4, 8, 5, 6, 1)
\]

We were able to prove that \( \varphi \) is in fact a bijection between \( \text{FR}_n \) and \( \text{UPF}_n \), allowing us to conclude that unit interval parking functions are enumerated by the Fubini Numbers. We also found a new formula for Fubini Numbers, shown in the corollary below.

Corollary

If \( n \geq 1 \), then \( \text{PF}_n \) is the set of all parking functions of length \( n \).

We then found and proved the following results for monotone UPFs:

Theorem

The number of unit interval parking functions (\( \text{UPF}_n \)) of length \( n \) is \( 2^n \).

Proof Sketch: We prove by induction, as each \( \alpha \in \text{UPF}_n \) has some prefix \( \beta \in \text{UPF}_{n-1} \), followed by \( n + 1 \) as \( n \rightarrow n + 1 \). Thus \( \text{UPF}_n = 2 \text{UPF}_{n-1} \).

Fubini Numbers

The r-Fubini numbers enumerate the number of ways to rank \( r \) players in a competition, allowing ties, such that a fixed \( r \) players are pairwise not tied with each other. Denote \( F_{b_n} \) as the \( n \)-th r-Fubini number.

We represent these numbers as combinatorial objects, we extended Hadaway’s Fubini rankings. We define an \( r \)-Fubini ranking of length \( n \) to be a Fubini ranking of length \( n \) such that the first \( r \) entries are distinct. Represent the set of \( r \)-Fubini rankings of length \( n \) as \( \text{PF}^r_n \).

We then found and proved the following theorem by finding a bijection between each set listed below and \( \text{PF}^r_n \).

Theorem

Each of the following is equal to \( \text{PF}^r_n \):

- The number of unit interval parking functions of length \( n + r - 1 \) with ties at indices \( \{2, 4, \ldots, 2r - 2\} \).
- The number of unit interval parking functions of length \( n + r \) with ties at indices \( \{1, 3, \ldots, 2r - 1\} \).

Proof Sketch: Restricting the domain of \( \varphi: \text{FR}_n \rightarrow \text{UPF}_n \) defined previously gives a bijection between \( \text{FR}_n \) and \( \text{UPF}_n \). Then we developed an “Add Tie Algorithm” to add a tie at a given index. Starting with \( \alpha \in \text{UPF}_n \), a specific sequence of executions of this algorithm produces a function \( \beta \in \text{UPF}_{n+1} \) of form (2), and one more execution at index 1 produces \( \gamma \in \text{UPF}_{n+2} \) of form (3).

Future Directions

An \( r \)-interval parking function is a parking function in which the displacement of each car is at most \( r \). We generated data for the number of \( r \)-interval parking functions for some values of \( r \) and \( n \) and found some interesting patterns, but we would like to find specific enumerations. We also want to further explore where and how frequently ties, ascents, and descents occur in UPFs.

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