Subsets of Unit Interval Parking Functions Enumerated by Fubini Numbers

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Parking Functions

Parking functions were first introduced by Konheim and Weiss in 1966 in the context of hashing problems. They can be defined as n-tuples $\alpha = (a_1, \ldots, a_n) \in [n]^n$ such that at least i entries of α are at most i for all $i \in [n]$. As the name suggests, parking functions can also be defined in terms of cars parking in the following arrangement:

Suppose n cars labeled $c_1, \ldots, c_n$ enter an empty linear parking lot with n spaces. Each car c_i has a preferred space $a_i \in [n]$. One by one and in order, each car first navigates to its preferred space and parks there if it is empty. If that space is already occupied, it parks in the next available spot, if one exists. If all n cars can park by this method, the n-tuple of the cars' preferences, $(a_1, \ldots, a_n)$, is called a **parking function** of length n. Denote by PF_n the set of all parking functions of length n.

> $(2, 2, 1)$ **is** a parking function: $\begin{matrix} c_3 & c_1 & c_2 \\ 1 & 2 & 3 \end{matrix}$
>
> $(2, 2, 3)$ **is not** a parking function: $\begin{matrix} \text{—} & c_1 & c_2 \\ 1 & 2 & 3 \end{matrix}$

Additional Definitions

- Let $\alpha = (a_1, \ldots, a_n)$ be a parking function. If car i parks in spot s_i, we call $s_i - a_i$ the **displacement** of car i.
- A **unit interval parking function** is a parking function in which the individual displacement of each car is at most 1. We refer to the set of unit interval parking functions of length n as UPF_n.

> $(2, 2, 1)$ **is** a unit interval parking function: $\begin{matrix} c_3 & c_1 & c_2 \\ 1 & 2 & 3 \end{matrix}$
>
> $(1, 1, 1)$ **is not** a unit interval parking function: $\begin{matrix} c_1 & c_2 & c_3 \\ 1 & 2 & 3 \end{matrix}$

- Let $\alpha = (a_1, \ldots, a_n)$ be a parking function. If $a_i = a_{i+1}$, we call index i a **tie**. If $a_i < a_{i+1}$, index i is an **ascent**. If $a_i > a_{i+1}$, index i is a **descent**.

> Index 1 is an **ascent** in the parking function $(1, 3, 2, 2)$.
> Index 2 is a **descent** in the parking function $(1, 3, 2, 2)$.
> Index 3 is a **tie** in the parking function $(1, 3, 2, 2)$.

Fubini Rankings and Fubini Numbers

A **Fubini ranking** is a sequence $\beta = (b_1, \ldots, b_n)$ that represents one way n players can rank in a competition, allowing ties. [1]. Formally, an n-tuple $\beta = (b_1, \ldots, b_n) \in [n]^n$ is a Fubini ranking of length n if $1 \in \beta$ and for all $i \in [n]$, if $i \in \beta$ and k entries of β are equal to i, then the next largest value in β is $i + k$. We denote FR_n as the set of all Fubini rankings of length n. The **Fubini numbers**, denoted Fb_n, enumerate Fubini rankings: $\mathrm{Fb}_n = |\mathrm{FR}_n|$.

> - $(1, 1, 1)$ **is** a Fubini ranking: all three players tied at rank 1.
> - $(1, 1, 2)$ **is not** Fubini ranking: two players tied at rank 1, so no one should place 2nd

Enumerations of Parking Functions and Unit Interval Parking Functions

Prior to this project, it was known that,

- The number of parking functions of length n is $(n + 1)^{n-1}$ [2].
- The weakly increasing (and weakly decreasing) parking functions are enumerated by the Catalan numbers [3].

We aimed to find similar enumerations for unit interval parking functions. In an unpublished paper, Hadaway (2022) found that unit interval parking functions are enumerated by the Fubini Numbers, to which we came up with an independent proof.

Definition

Define $\varphi : \mathrm{FR}_n \mapsto \mathrm{UPF}_n$ as follows: Given $\beta = (b_1, \ldots, b_n) \in \mathrm{FR}_n$, let $\varphi(\beta) = (a_1, \ldots, a_n) \in \mathrm{UPF}_n$, where

$$a_i = \begin{cases} b_i & \text{if index } i \text{ is the first or second occurrence of } b_i \\ b_i + k - 2 & \text{if index } i \text{ is the } k^{\text{th}} \text{ occurrence of } b_i \ (k > 2) \end{cases}.$$

> $\beta = (4, 3, 1, 4, 8, 4, 4, 1) \quad \rightarrow \quad \varphi(\beta) = (4, 3, 1, 4, 8, 5, 6, 1)$

We were able to prove that φ is in fact a bijection between FR_n and UPF_n, allowing us to conclude that unit interval parking functions are enumerated by the Fubini numbers. We also found a new formula for Fubini Numbers, shown in the corollary below.

Corollary

If $n \geq 1$ and $(c_1, c_2, \ldots, c_k) \vDash n$ denotes a composition of n, then

$$Fb_n = \sum_{k=1}^{n} \left(\sum_{(c_1, c_2, \ldots, c_k) \vDash n} \binom{n}{c_1, c_2, \ldots, c_k} \right).$$

We then found and proved the following results for monotone UPFs:

Theorem

The number of weakly increasing unit interval parking functions ($\mathrm{UPF}_n^{\uparrow}$) of length n is 2^{n-1}.

Proof Sketch: We prove by induction, as each $\alpha \in \mathrm{UPF}_n^{\uparrow}$ has some prefix $\beta \in \mathrm{UPF}_{n-1}^{\uparrow}$ followed by n or $n - 1$. Thus $|\mathrm{UPF}_n^{\uparrow}| = 2|\mathrm{UPF}_{n-1}^{\uparrow}|$.

Theorem

The number of weakly decreasing unit interval parking functions ($\mathrm{UPF}_n^{\downarrow}$) of length n is F_{n+1} (where F_n is the Fibonacci sequence: $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$.)

Proof Sketch: We prove by induction, noting that $\alpha \in \mathrm{UPF}_n^{\downarrow}$ has one of two forms:

1. $\alpha = (n, a_2, a_3, \ldots, a_n)$, where $(a_2, \ldots, a_n) \in \mathrm{UPF}_{n-1}^{\downarrow}$, or
2. $\alpha = (n-1, n-1, a_3, a_4, \ldots, a_n)$, where $(a_3, \ldots, a_n) \in \mathrm{UPF}_{n-2}^{\downarrow}$.

Thus $|\mathrm{UPF}_n^{\downarrow}| = |\mathrm{UPF}_{n-1}^{\downarrow}| + |\mathrm{UPF}_{n-2}^{\downarrow}|$, which is the Fibonacci recurrence.

r-Fubini Numbers

The r**-Fubini numbers** enumerate the number of ways to rank players in a competition, allowing ties, such that a fixed r players are pairwise not tied with each other. Denote Fb_n^r as the n^{th} r-Fubini number.

To represent these numbers as combinatorial objects, we extended Hadaway's Fubini rankings. We define an r**-Fubini ranking** of length n to be a Fubini ranking of length n such that the first r elements are distinct. Represent the set of r-Fubini rankings of length n as FR_n^r. Then $\mathrm{Fb}_n^r = |\mathrm{FR}_n^r|$. Note that $\mathrm{Fb}_n^0 = \mathrm{Fb}_n^1 = \mathrm{Fb}_n$, while $\mathrm{Fb}_n^n = n!$.

> $(3, 1, 2, 3)$ **is** a 3-Fubini ranking: The first three players are not tied.
> $(3, 3, 1, 2)$ **is** a Fubini ranking, but **is not** a 3-Fubini ranking: the first two players are tied.

r-Fubini Bijection

Next we extended this UPF-Fubini connection to include to r-Fubini rankings. To do so, we defined r**-unit interval parking functions**, denoted UPF_n^r, as the set of unit interval parking functions in which the first r cars all have distinct preferences. We then proved the following theorem by finding a bijection between each set listed below and FR_n^r.

Theorem

Each of the following is equal to Fb_n^r :

1. UPF_n^r
2. The number of unit interval parking functions of length $n + r - 1$ with ties at indices $\{2, 4, \ldots, 2r - 2\}$.
3. The number of unit interval parking functions of length $n + r$ with ties at indices $\{1, 3, \ldots, 2r - 1\}$.

Proof Sketch: Restricting the domain of $\varphi : \mathrm{FR}_n \mapsto \mathrm{UPF}_n$, defined previously, gives a bijection between FR_n^r and UPF_n^r. We then developed an "Add Tie Algorithm" to add a tie at a given index. Starting with $\alpha \in \mathrm{UPF}_n^r$, a specific sequence of executions of this algorithm produces a function $\beta \in \mathrm{UPF}_{n+r-1}$ of form (2), and one more execution at index 1 produces $\gamma \in \mathrm{UPF}_{n+r}$ of form (3).

Future Directions

An ℓ-interval parking function is a parking function in which the displacement of each car is at most ℓ. We generated data for the number of ℓ-interval parking functions for some values of ℓ and n and found some interesting patterns, but we would like to find specific enumerations. We also want to further explore where and how frequently ties, ascents, and descents occur in UPFs.

References/Acknowledgements

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This material is based upon work supported by the National Science Foundation under Grant No. DMS-1929284 while the authors were in residence at the Institute for Computational and Experimental Research in Mathematics in Providence, RI, during the Summer@ICERM program.